Lesson 22: Congruence Criteria for Triangles—SAS

Classwork

Opening Exercise

Answer the following question. Then discuss your answer with a partner.

Is it possible to know that there is a rigid motion that takes one triangle to another without actually showing the particular motion?

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Discussion

It is true that we will not need to show the rigid motion to be able to know that there is one. We are going to show that there are criteria that refer to a few parts of the two triangles and a correspondence between them that guarantee congruency (i.e., existence of rigid motion). We start with the Side-Angle-Side (SAS) criteria.

**Side-Angle-Side triangle congruence criteria (SAS):** Given two triangles $△ABC$ and $△A'B'C'$ so that $AB=A'B'$ (Side), $∠A=∠A'$ (Angle), $AC=A^{'}C^{'}$ (Side). Then the triangles are congruent.

The steps below show the most general case for determining a congruence between two triangles that satisfy the SAS criteria. Note that not all steps are needed for every pair of triangles. For example, sometimes the triangles will already share a vertex. Sometimes a reflection will be needed, sometimes not. What is important is that we can always use the steps below—some or all of them—to determine a congruence between the two triangles that satisfies the SAS criteria.

**Proof:** Provided the two distinct triangles below, assume $AB=A'B'$ (Side), $∠A=∠A'$ (Angle), $AC=A'C'$ (Side).



By our definition of congruence, we will have to find a composition of rigid motions will map $△A'B'C'$ to $△ABC$. So we must find a congruence $F$ so that $F(△A^{'}B^{'}C^{'})$ = $△ABC.$ First, use a translation $T$ to map a common vertex.

Which two points determine the appropriate vector?

Can any other pair of points be used? \_\_\_\_\_\_\_\_ Why or why not? \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

State the vector in the picture below that can be used to translate $△A'B'C'$: \_\_\_\_\_\_\_\_\_\_\_\_\_

Using a dotted line, draw an intermediate position of $△A'B'C'$ as it moves along the vector:

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****After the translation (below), $T\_{\rightharpoonaccent{A'A}}(△A^{'}B^{'}C^{'})$ shares one vertex with $△ABC$, $A$. In fact, we can say
$T\_{\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_}\left(△\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\right)= △\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_$.

Next, use a clockwise rotation $R\_{∠CAC''}$ to bring the sides AC” to AC (or counterclockwise rotation to bring AB” to AB).

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A rotation of appropriate measure will map $\rightharpoonaccent{AC"}$ to $\rightharpoonaccent{AC}$, but how can we be sure that vertex $C"$ maps to $C$? Recall that part of our assumption is that the lengths of sides in question are equal, ensuring that the rotation maps $C''$ to $C$. $(AC=AC'',$ the translation performed is a rigid motion, and thereby did not alter the length when $AC'$ became $AC^{''}.)$

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After the clockwise rotation, $R\_{∠CAC''}(△AB''C'')$, a total of two vertices are shared with $△ABC$, $A$ and $C$. Therefore,
$R\\_\\_\\_\\_\\_\\_\\_\\_(△\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_)= △\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_\\_$.

Finally, if $B'''$ and $B$ are on opposite sides of the line that joins $AC$, a reflection $Λ\_{AC}$ brings $B'''$ to the same side as $B$.



Since a reflection is a rigid motion and it preserves angle measures, we know that $∠B^{'''}AC=∠BAC$ and so $\rightharpoonaccent{AB'''}$ maps to $\rightharpoonaccent{AB}$. If, however, $\rightharpoonaccent{AB'''}$ coincides with $\rightharpoonaccent{AB}$, can we be certain that $B'''$ actually maps to $B$? We can, because not only are we certain that the rays coincide, but also by our assumption that $AB=AB'''$.(Our assumption began as $AB=A^{'}B^{'}$, but the translation and rotation have preserved this length now as $AB'''$.) Taken together, these two pieces of information ensure that the reflection over $AC$ brings $B'''$ to $B$.

Another way to visually confirm this is to draw the marks of the construction for $AC$.

Write the transformations used to correctly notate the congruence (the composition of transformations) that takes $△A^{'}B^{'}C^{'}≅ △ABC$:

 $F$ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

 $G$ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

 $H$ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

$\\_\\_\\_\\_(\\_\\_\\_\\_(\\_\\_\\_\\_(△A^{'}B^{'}C^{'})$ $=$ $△ABC$

We have now shown a sequence of rigid motions that takes $△A'B'C'$ to $△ABC$ with the use of just three criteria from each triangle: two sides and an included angle. Given *any* two, distinct, triangles, we could perform a similar proof. There are other situations, where the triangles are not distinct, where a modified proof will be needed to show that the triangles map onto each other. Examine these below.

Example 1

What if we had the SAS criteria for two triangles that were not distinct? Consider the following two cases. How would the transformations needed to demonstrate congruence change?

|  |  |  |
| --- | --- | --- |
| **Case** | **Diagram** | **Transformations Needed** |
| **Shared Side** |  |  |
| **Shared Vertex** |  |  |

Exercises 1–3

1. Given: Triangles with a pair of corresponding sides of equal length and a pair of included angles of equal measure. Sketch and label three phases of the sequence of rigid motions that prove the two triangles to be congruent.



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*Directions:* Justify whether the triangles meet the SAS congruence criteria; explicitly state which pairs of sides or angles are congruent and why. If the triangles do meet the SAS congruence criteria, describe the rigid motion(s) that would map one triangle onto the other.



1. Given: $∠LMN=∠LNO, MN=OM$.

Do $△LMN$ and $△LOM$ meet the SAS criteria?

1. Given: $∠HGI=∠JIG, HG=JI$.

Do $△HGI$ and $△JIG$ meet the SAS criteria?

Problem Set

*Directions:* Justify whether the triangles meet the SAS congruence criteria; explicitly state which pairs of sides or angles are congruent and why. If the triangles do meet the SAS congruence criteria, describe the rigid motion(s) that would map one triangle onto the other.

1. Given: $AB∥CD, AB=CD$.

Do $△ABD$ and $△CDB$ meet the SAS criteria?

1. Given: $∠R=25°, RT=7", SU=5", ST=5"$*.*

Do $△RSU$ and $△RST$ meet the SAS criteria?

1. Given: $ KM$ and $JN$ bisect each other.

Do $△JKL$ and $△NML$ meet the SAS criteria?



1. Given: $∠1=∠2, BC=DC$.

Do $△ABC$ and $△ADC$ meet the SAS criteria?

1. Given: $AE$ bisects angle$ ∠BCD, BC=DC$.

Do $△CAB$ and $△CAD$ meet the SAS criteria?



1. Given: $SU $and $RT $bisect each other.

Do $△SVR$ and $△UVT$ meet the SAS criteria?

1. Given: $JM=KL, \overbar{ JM}⊥\overbar{ML}, \overbar{ KL}⊥\overbar{ML}$.

Do $△JML$ and $△KLM$ meet the SAS criteria?

1. Given: $\overbar{BF}⊥\overbar{AC}, \overbar{CE}⊥\overbar{AB}$.

Do $△BED$ and $△CFD$ meet the SAS criteria?

1. Given:$ ∠VXY=∠VYX$.

Do $△VXW$ and $△VYZ$ meet the SAS criteria?



1. Given:$ △RST$ is isosceles, $SY=TZ$.

Do $△RSY$ and $△RTZ$ meet the SAS criteria?